Properties and Natural Extensions of p-adic β-shifts

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Interval β shifts

Let $\beta \in \mathbb{R}$ with $\beta > 1$. Define

$$\begin{aligned} T_\beta : I &\to I \\ x &\mapsto \{\beta x\}. \end{aligned}$$

For $x \in I$, we can use T_β to find the β expansion

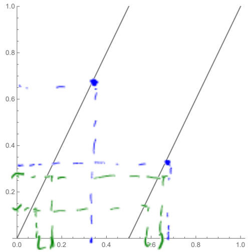
$$x = \sum_{k=1}^{\infty} \frac{d_k}{\beta^k}, \text{ where } d_k = \lfloor \beta T_\beta^{k-1}(x) \rfloor.$$

Interval β shifts

$$T_2 : I \to I$$
$$x \mapsto \{2x\}$$

$x = 1/3$

$\beta x = 2/3$



$d_1 = 0$ $\quad T_2(1/3) = \frac{2}{3}$

$\beta T_2(\frac{1}{3}) = \frac{4}{3} = 1 + 1/3$

$d_2 = 1$ $\quad T_2^2(1/3) = 1/3$

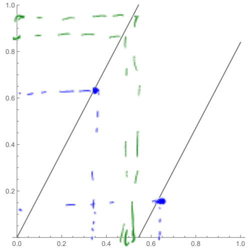
$$\sum_{k=1}^{\infty} \frac{d_k}{\beta^k} = \frac{0}{2} + \frac{1}{2^2} + \frac{0}{2^3} + \frac{1}{2^4} + \cdots$$

$0101\ldots = .\overline{01}$

Interval β shifts

$$T_\beta : I \to I$$
$$x \mapsto \{\beta x\}$$
$$\beta^3 - \beta^2 - \beta - 1 = 0, \; \beta = 1.83929\ldots$$

$x = 1/3$



$\beta \frac{1}{3} \approx 0.6131$

$d_1 = 0$ $\quad T_\beta(1/3) \approx 0.6131$

$\beta T_\beta(1/3) \approx 1.1277$

$d_2 = 1$ $\quad T_\beta^2(1/3) \approx 0.1277$

Interval β shifts

$$\beta T_\beta^2(1/3) \approx 0.2348$$

$$d_3 = 0 \qquad T_\beta^3(1/3) \approx 0.2348$$

$$\frac{0}{2} + \frac{1}{2^2} + \frac{0}{2^3} + \frac{0}{2^4} + \cdots$$

$$.\overline{0100010110000}$$

Interval β shifts

If $\beta > 1$ is an integer, then

- T_β preserves Lebesgue measure and
- T_β is isomorphic to the one-sided shift on β symbols.

If $\beta > 1$ is not an integer, then

- T_β preserves an absolutely continuous probability measure and
- T_β is weakly Bernoulli.

In both cases, the entropy of T_β is $\log(\beta)$.

The p-adic numbers

Let p be a prime number.

p-adic numbers:

$$\mathbb{Q}_p = \left\{ \sum_{i=k}^{\infty} x_i p^i : k, x_i \in \mathbb{Z} \text{ and } 0 \leqslant x_i < p \text{ for all } i \geqslant k \right\}$$

p-adic integers:

$$\mathbb{Z}_p = \left\{ \sum_{i=0}^{\infty} x_i p^i : 0 \leqslant x_i < p \text{ for all } i \geqslant 0 \right\}$$

p-adic absolute value:

$$|x|_p = \begin{cases} p^{-\operatorname{ord}_p(x)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

If $x = \sum_{i=k}^{\infty} x_i p^i \in \mathbb{Q}_p$, then let

- $\lfloor x \rfloor_p = \sum_{i=0}^{\infty} x_i p^i$ be the p-adic integer part and
- $\{x\}_p = \sum_{i=k}^{-1} x_i p^i$ be the p-adic fractional part.

Let $\beta \in \mathbb{Q}_p$ with $|\beta|_p > 1$. Define

$$\begin{aligned} S_\beta : \mathbb{Z}_p &\to \mathbb{Z}_p \\ x &\mapsto \lfloor \beta x \rfloor_p. \end{aligned}$$

For $x \in \mathbb{Z}_p$, we can use S_β to find the β expansion

$$x = \sum_{k=1}^{\infty} \frac{d_k}{\beta^k}, \text{ where } d_k = \{\beta S_\beta^{k-1}(x)\}_p.$$

p-adic β shifts

$p = 2$

$$S_{1/2} : \mathbb{Z}_p \to \mathbb{Z}_p$$
$$x \mapsto \left\lfloor \frac{1}{2} x \right\rfloor_p$$

$$x = 1/3 = 1 + 2 + 0 \cdot 2^2 + 2^3 + 2^5 + \cdots \quad (1\ 1\ 0\ 1\ 0\ 1 \cdots)$$

$$\beta x = \frac{1}{2} + 1 + 2^2 + 2^4 + 2^6 + \cdots$$

$$d_1 = \frac{1}{2}$$

$$S_{1/2}(1/3) = 1 + 2^2 + 2^4 + 2^6 + \cdots \quad (1\ 0\ 1\ 0\ 1\ 0\ 1 \cdots)$$

$$\beta S_{1/2}(1/3) = 1/2 + 2 + 2^3 + 2^5 + \cdots$$

$$d_2 = 1/2$$

$$S_{1/2}^2(1/3) = 2 + 2^3 + 2^5 + \cdots \quad (0\ 1\ 0\ 1\ 0\ 1 \cdots)$$

$$\beta S^3_{1/2}(1/3) = 1 + 2^2 + 2^4 + 2^6 + \cdots$$

$$S^3_{1/2}(1/3) =$$

$$d_2 = 0$$

$$\sum_{k=1}^{\infty} \frac{d_k}{\beta^k} = \frac{1/2}{1/2} + \frac{\frac{1}{2}}{1/2^2} + \frac{0}{1/2^3} + \frac{1/2}{1/2^4} + \frac{0}{1/2^5} + \cdots$$

$$= 1 + 2 + 2^3 + 2^5 + \cdots$$

$$.\tfrac{1}{2}\overline{\tfrac{1}{2}\,0}$$

p-adic β shifts

$p=2$

$$S_\beta : \mathbb{Z}_p \to \mathbb{Z}_p$$
$$x \mapsto \lfloor \beta x \rfloor_p$$

$$\beta^2 + \frac{1}{2}\beta + \frac{1}{2} = 0, \;\; \beta = \frac{1}{2} + 2 + 2^4 + 2^6 + 2^7 + 2^8 + \cdots$$

$x = 1/3$

βx

$d_1 = 1/2$ $\qquad S_\beta(1/3) = 1 + 2 + 2^3 + 2^4 + \ldots$

$\beta S_\beta(1/3)$

$d_2 = 1/2$ $\qquad S_\beta^2(1/3) = 1 + 2 + 2^4 + 2^5 + 2^6 + \ldots$

$\beta S_\beta^2(1/3)$

p-adic β shifts

$d_3 = \frac{1}{2}$ $\qquad S_\beta^3(\frac{1}{3}) = 1 + 2 + 2^2 + 2^3 + \cdots$

$$. \frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\overline{\frac{1}{2}\,\frac{1}{2}\,0\,0\,\frac{1}{2}\,0\,\frac{1}{2}\,0}$$

Let $\beta = 1/p \in \mathbb{Q}_p$. Then

$$\begin{aligned} S_{1/p}(x) &= \left\lfloor \frac{1}{p}(x_0 + x_1 p + x_2 p^2 + x_3 p^3 + \cdots) \right\rfloor_p \\ &= \left\lfloor x_0 \frac{1}{p} + x_1 + x_2 p + x_3 p^2 + \cdots \right\rfloor_p \\ &= x_1 + x_2 p + x_3 p^2 + \cdots \\ &= \sigma(x). \end{aligned}$$

From the last slide,

- $S_{1/p}(x) = \sigma(x)$

p-adic β shifts

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For integers $n \geqslant 1$,

- $S_{1/p^n}(x) = \sigma^n(x)$.

p-adic β shifts

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- $S_{1/p}(x) = \sigma(x)$

For integers $n \geqslant 1$,

- $S_{1/p^n}(x) = \sigma^n(x)$.

If $\beta \in \mathbb{Q}_p$ with $|\beta|_p = p^n > 1$, then there exists $b \in \mathbb{Z}_p$ with $|b|_p = 1$ such that $\beta = b/p^n$.

- $S_\beta(x) = \sigma^n(bx)$.

Let $|\beta|_p > 1$.

d'Ambros, Everest, Miles, Ward, 2000

- The topological entropy of the p-adic β-shift is $\log|\beta|_p$
- With respect to Haar measure, the p-adic β-shift is invariant and ergodic.

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- If $|\beta|_p = p^n$, then S_β is topologically and measurably isomorphic to σ^n.

Scheicher, Sirvent, Surer, 2015

- Connect the arithmetic properties of β to the arithmetic properties of points that have a finite or periodic β expansion.

Natural extensions

Definition

A measure-preserving dynamical system $(Y, \mathcal{C}, \nu, S)$ is a **factor** of $(X, \mathcal{F}, \mu, T)$ if there exists $\psi : X \to Y$ that is measurable, surjective, and satisfies

1. $\psi^{-1}\mathcal{C} \subset \mathcal{F}$,
2. $\psi \circ T = S \circ \psi$, and
3. $\mu(\psi^{-1}E) = \nu(E)$ for all $E \in \mathcal{C}$.

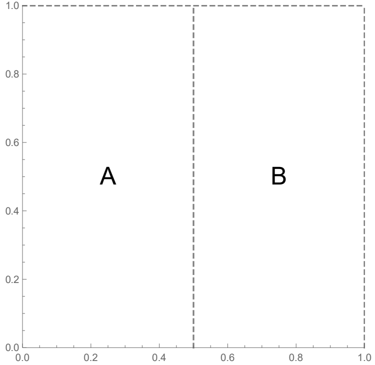
Definition

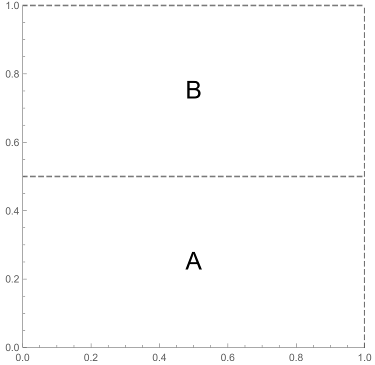
An invertible measure-preserving dynamical system $(X, \mathcal{F}, \mu, T)$ is a **natural extension** of $(Y, \mathcal{C}, \nu, S)$ if Y is a factor of X and $\bigvee_{n=0}^{\infty} T^n\psi^{-1}\mathcal{C} = \mathcal{F}$.

Dajani, Kraaikamp, Solomyak, 1996.

$$\mathcal{T}_\beta(x,y) = \left(\{\beta x\}, \frac{1}{\beta}(\lfloor \beta x \rfloor + y)\right)$$

$$\beta = 2$$

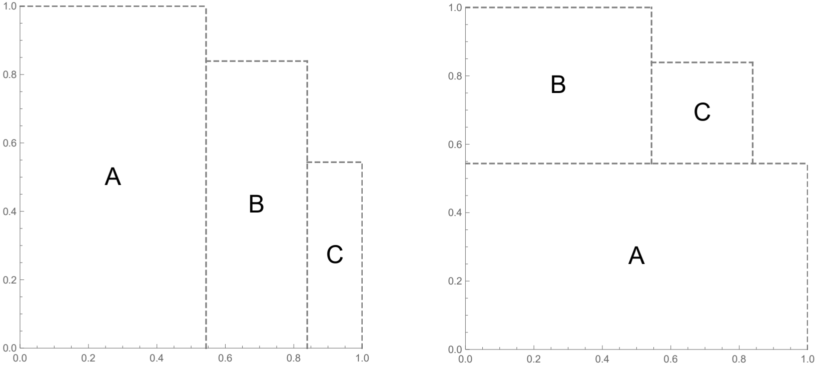




Dajani, Kraaikamp, Solomyak, 1996.

$$\mathcal{T}_\beta(x,y) = \left(\{\beta x\}\,, \frac{1}{\beta}(\lfloor \beta x \rfloor + y) \right)$$

$$\beta^3 - \beta^2 - \beta - 1 = 0,\ \beta \approx 1.83929$$



Natural extensions for β-shifts

Dajani, Kraaikamp, Solomyak, 1996.

For the interval:

$$\mathcal{T}_\beta(x,y) = \left(\{\beta x\}, \tfrac{1}{\beta}(\lfloor \beta x \rfloor + y)\right)$$

For the p-adic integers:

$$\mathcal{S}_\beta(x,y) = \Big(\underbrace{\lfloor \beta x \rfloor_p}_{u}, \underbrace{\tfrac{1}{\beta}(\{\beta x\}_p + y)}_{v}\Big)$$

$$\mathcal{S}_\beta^{-1}(u,v) = \left(\tfrac{1}{\beta}(\{\beta v\}_p + u), \lfloor \beta v \rfloor_p, \right)$$

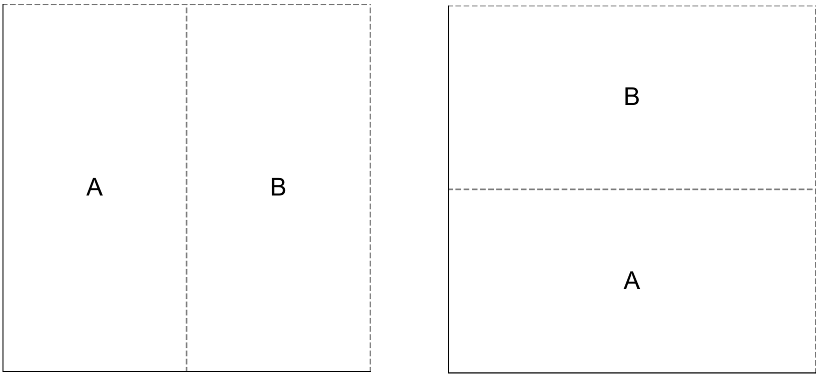
Key idea: $\{\beta x\}_p = \{\beta v\}_p$

$\beta v = \{\beta x\}_p + y$ (in $\mathbb{Z}_p$)

$\{\beta v\}_p = \{\beta x\}_p$

Consider the natural extension of $S_{1/2}$ on $\mathbb{Z}_2$.

$$\mathcal{S}_{1/2}(x,y) = \left(\left\lfloor \frac{1}{2}x \right\rfloor_2, 2\left(\left\{ \frac{1}{2}x \right\}_2 + y \right) \right)$$

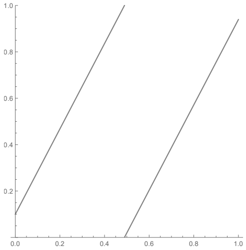


Interval α, β shifts

Let $0 < \alpha < 1$ and $\beta > 1$. Define

$$\begin{aligned} T_{\alpha,\beta} : I &\to I \\ x &\mapsto \{\alpha + \beta x\}. \end{aligned}$$

$$\alpha = 0.1,\ \beta^3 - \beta^2 - \beta - 1 = 0,\ \beta \approx 1.83929$$



p-adic α, β shifts

Let $\alpha, \beta \in \mathbb{Q}_p$ with $|\beta|_p = p^n > 1$. Define

$$\begin{aligned} S_{\alpha,\beta} : \mathbb{Z}_p &\to \mathbb{Z}_p \\ x &\mapsto \lfloor \alpha + \beta x \rfloor_p. \end{aligned}$$

As before, we can find $a, b \in \mathbb{Z}_p$ with $|b|_p = 1$ such that $S_{\alpha,\beta}(x) = \sigma^n(a + bx)$. If $|x - y|_p \leqslant 1/p^n$, then

$$\begin{aligned} |S_{\alpha,\beta}(x) - S_{\alpha,\beta}(y)|_p &= |\sigma^n(a + bx) - \sigma^n(a + by)|_p \\ &= p^n|(a + bx) - (a + by)|_p \\ &= p^n|x - y|_p. \end{aligned}$$

Locally scaling transformations

Kingsbery, Levin, Preygel, Silva, 2011

Definition

A transformation $S : \mathbb{Z}_p \to \mathbb{Z}_p$ is (p^{-n}, p^n)-locally scaling map if $|x - y|_p \leqslant p^{-n}$ implies that $|S(x) - S(y)|_p = p^n|x - y|_p$.

Theorem (Kingsbery et al. 2011)

Let $S : \mathbb{Z}_p \to \mathbb{Z}_p$ be a (p^{-n}, p^n)-locally scaling map. Then S is topologically and measurably isomorphic to σ^n.

Thus, if $|\beta|_p = p^n$, then $S_{\alpha,\beta}$ is isomorphic to σ^n.

Structure theorem

Theorem (F. 2019)

The transformation $S : \mathbb{Z}_p \to \mathbb{Z}_p$ is a (p^{-n}, p^n)-locally scaling map if, and only if, there exists an isometry f on $\mathbb{Z}_p$ such that $S = \sigma^n \circ f$.

Idea: For $x = \sum_{i=0}^{\infty} x_i p^i$ in $\mathbb{Z}_p$, take $f(x) = p^n S(x) + \sum_{i=0}^{n-1} x_i p^i$.

Natural extension for scaling maps

Theorem (F. 2019)

Suppose that S *on* $(\mathbb{Z}_p, \mathcal{B}, \lambda)$ *satisfies* $S = \sigma^n \circ f$ *for some integer* $n \geqslant 1$ *and an isometry* f *on* $(\mathbb{Z}_p, |\cdot|_p)$*. Then the natural extension of* $(\mathbb{Z}_p, \mathcal{B}, \lambda, S)$ *is* $(\mathbb{Z}_p \times \mathbb{Z}_p, \mathcal{B} \times \mathcal{B}, \lambda \times \lambda, \mathcal{S})$*, where*

$$\begin{aligned} \mathcal{S} : \mathbb{Z}_p \times \mathbb{Z}_p &\to \mathbb{Z}_p \times \mathbb{Z}_p \\ (x, y) &\mapsto \left(\left\lfloor \frac{f(x)}{p^n} \right\rfloor_p, f^{-1}\left(p^n \left(\left\{ \frac{f(x)}{p^n} \right\}_p + y \right) \right) \right) \end{aligned}$$

Using the f from the proof of the structure theorem:

$$\begin{aligned} \mathcal{S} : \mathbb{Z}_p \times \mathbb{Z}_p &\to \mathbb{Z}_p \times \mathbb{Z}_p \\ (x, y) &\mapsto \left(S(x), f^{-1}\left(p^n y + \sum_{i=0}^{n-1} x_i p^i \right) \right) \end{aligned}$$

Other measures

Ball of radius p^n centered at a:

$$B_{p^n}(a) = \{x \in \mathbb{Z}_p : |x - a|_p \leqslant p^n\}.$$

i.i.d. product measures:

$$\mu = \prod_{i=0}^{\infty} \{q_0, q_1, \ldots, q_{p-1}\}$$

$$\mu\left(B_{p^{-n}}\left(\sum_{i=0}^{n-1} a_i p^i\right)\right) = \prod_{i=0}^{n-1} q_{a_i}.$$

Other measures

Multiplication maps:

$$\begin{aligned} M_b : \mathbb{Z}_p &\to \mathbb{Z}_p \\ x &\mapsto bx \end{aligned}$$

Theorem (F.)

For an i.i.d. product measure μ on $\mathbb{Z}_p$ defined by a probability vector $(q_0, q_1, \ldots q_{p-1})$, the multiplication M_{-1} is nonsingular with respect to μ if and only if the probability vector is palindromic.

Moreover, if $b \in \mathbb{Z}_p$ is a rational number with $|b|_p = 1$ and $b \neq \pm 1$, then the multiplication M_b is nonsingular with respect to μ if and only if μ is Haar measure.

Theorem (F.)

Let μ be an i.i.d. product measure on $\mathbb{Z}_p$ defined by a probability vector $(q_0, q_1, \ldots q_{p-1})$.

If $\beta = 1/p^n$ for some integer $n > 1$, then T_β preserves μ.

If $\beta = -1/p^n$ for some integer $n > 1$, then T_β is nonsingular with respect to μ if and only if the probability vector is palindromic.

If $\beta = b/p^n$, where $b \in \mathbb{Z}_p$ is a rational number with $|b|_p = 1$ and $b \neq \pm 1$, then T_β is nonsingular with respect to μ if and only if μ is Haar measure.